On Tightness of Constraints*

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Abstract. The *tightness* of a constraint refers to how restricted the constraint is. The existing work shows that there exists a relationship between tightness and global consistency of a constraint network. In this paper, we conduct a comprehensive study on this relationship. Under the concept of k-consistency (k is a number), we strengthen the existing results by establishing that only some of the tightest, *not all*, binary constraints are used to predict a number k such that strong k-consistency ensures global consistency of an arbitrary constraint network which may include non-binary constraints. More importantly, we have identified a lower bound of the number of the tightest constraints we *have to* consider in predicting the number k. To make better use of the tightness of constraints, we propose a new type of consistency: *dually adaptive consistency*. Under this concept, only the tightest *directionally relevant* constraint on each variable (and thus in total $n-1$ such constraints where n is the number of variables) will be used to predict the level of "consistency" ensuring global consistency of a network.

1 Introduction

Informally, the *tightness* of a binary constraint is the *maximum number of compatible values* allowed for each value of the constrained variables. For example, let $x, y \in 1..10$ be two variables and consider the constraint $x = y$. For any value of x, the constraint allows at most one compatible value for y. The constraint is also said 1-tight. There is a very interesting relationship between the tightness and the global consistency of a constraint network. (When we say a network is *globally consistent*, we mean it is satisfiable.) For example, if all the constraints in a binary network is 1-tight, path consistency (i.e., strongly 3-consistency) is sufficient to determine the global consistency of the network. If *not all* constraints are 1-tight, the existing method will use the least tight constraint to determine the level of consistency sufficient for global consistency. *This level is higher (and thus more expensive) if the constraints are less tight.* The motivation of this paper is to determine the level of consistency by using less number of and possibly tighter constraints. For example, by our results, if a constraint

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network with n variables has n 1-tight constraints on "correct" variables, a local consistency similar to path consistency is able to ensure its global consistency.

2 Tightness under k-consistency

In this section, we show that we can use a smaller number of m-tight constraint to determine the weak m-tightness of a constraint network and thus the local level of consistency ensuring the global consistency of the network. On the other hand, we identify a lower bound on the number of m-tight constraints in a weakly m-tight network.

In this paper, n denotes the number of variables in a constraint network. A constraint is represented as c_S where S is the set of variables involved in the constraint.

m-tightness [4] Given a number m, a constraint c_S is *m-tight on x* if and only if any instantiation of $S - \{x\}$ is compatible with all or at most m values of x. If a constraint is m-tight on x, its tightness on x is m. A constraint is m-tight if it is m-tight on each of its variables.

Relevant Constraints A *relevant constraint* on a variable x with respect to a set of variables Y is one whose scope consists of only x and variables from Y. In other words, it involves x, but does not involve any variable outside Y. $R_Y(x)$ is used to denote the set of relevant constraints on x wrt Y. When Y is clear from the context, $R(x)$, rather than $R_Y(x)$, will be used.

Weakly m-tight Constraint Networks A constraint network is *weakly m-tight* at level k iff for every set of variables $Y = \{x_1, \cdots, x_k\}$ and a new variable x, there exists an m-tight relevant constraint on x wrt Y.

This definition is simpler than the one given in [5] where every set Y of size k or greater than k is considered. The Proposition 1 below shows that the two definitions are equivalent.

Remark The definition needs the assumption that given a network, there is a universal constraint among any set of variables on which there is no explicit constraint. A universal constraint on a set of variables allows any instantiation of the variables. In this section, we need to keep in mind that there is a constraint among any set of variables.

For a weakly tight network, we have this consistency result.

Theorem 1. [5] *If for some m, a constraint network with constraints of arity at most r is strongly $((m+1)(r-1)+1)$-consistent and weakly m-tight at level $((m+1)(r-1)+1)$, it is strongly n-consistent.*

There is a strong relationship among different levels of weak tightness in a network.

Proposition 1. *If a constraint network is weakly m-tight at level k for some m, it is weakly m-tight at any level $j > k$.*

The following result gives a sufficient condition for a constraint network to be weakly m-tight at level k.

Theorem 2. *A constraint network is weakly m-tight at level k if for every variable in the network, there are at least $n - k + 1$ m-tight binary constraints on it for some m and k.*

The next result shows a lower bound on the number of m-tight constraints in a network weakly tight at level 3.

Theorem 3. *For a constraint network to be weakly m-tight at level 3, it needs at least $n(n-1)/2 - 2\lfloor n/3 \rfloor$ when $n = 0, 1 (\texttt{mod}\ 3)$, or $(n-2)(3n-1)/6$ when $n = 2 (\texttt{mod}\ 3)$, m-tight binary or ternary constraints.*

3 Tightness under directional consistency

From the results in the previous section, under the concept of k-consistency we can not reduce, by much, the number of m-tight constraints required to predict the k-consistency ensuring global consistency. In this section, we examine tightness under directional consistency [2].

Directionally Relevant Constraints Given an ordering of variables, a relevant constraint on x with respect to a set of variables Y is *directionally relevant* if it involves x and only variables before x.

Definition 1. *A constraint network is* directionally *weakly m-tight at level k with respect to an order of variables iff for every set of variables $Y = \{x_1, \cdots, x_l\}$ ($l : k..n-1$) and a new variable x , there exists an m-tight directionally relevant constraint on x.*

Directional weak m-tightness does not require a constraint to be tight on *each* of its variables. It is related to global consistency in the following way.

Theorem 4. *Given a network, let r be the maximum arity of its constraints. If it is directionally weakly m-tight at level $(m+1)(r-1)+1$ and is strongly directionally $(m+1)(r-1)+1$-consistent, then it is strongly directionally n-consistent.*

The next result presents a sufficient condition for a network to be directionally weakly m-tight.

Theorem 5. *A network of arbitrary constraints is directionally weakly m-tight at level k with respect to a variable ordering if for all $i > k$, there are at least $i - k$ directionally relevant binary constraints which are m-tight on the i_{th} variable.*

4 Dually adaptive consistency

One main purpose of our characterization of weakly m-tight network is to help identify a consistency condition under which a solution of a network can be found without backtracking (i.e., efficiently).

Motivated by the idea of adaptive consistency [2], we propose a concept of dually adaptive consistency which makes use of both topological structure and the semantics of a constraint network. In the following definition, the *width* of a variable with respect to a variable ordering is the number of the directionally relevant constraints on it.

Given a network, a variable ordering, and a variable x, let $DR(x)$ be the set of directionally relevant constraints on x and S be the union of the scopes of the constraints of $DR(x)$. The constraints of $DR(x)$ are *consistent* on x, if and only if for any consistent instantiation $\bar{a}$ of $S - \{x\}$, there exists $u \in D_x$ such that $(\bar{a}, u)$ satisfies all the constraints in $DR(x)$.

Definition 2. *Given a constraint network and an ordering of its variables, let* c_x *be one of the tightest directionally relevant constraints on* x *and* m_x *be its tightness. It is* dually adaptively consistent *if and only if*

1) for any variable x *whose width is not greater than* m_x*, the directionally relevant constraints on it are consistent, and*

2) for any variable x *whose width is greater than* m_x*,* c_x *is consistent with every other* m_x *directionally relevant constraints on* x*.*

Lemma 1. *Given a number* m *and a collection of sets* $\{E_1, \cdots, E_l\}$*, assume there is a set* E *among them such that* $|E| \leq m$*.* $\bigcap_{i \in 1..l} E_i \neq \emptyset$ *iff the intersection of* E *and every other* m *sets is not empty.*

This lemma results in the result that a dually adaptive consistent constraint network is globally consistent.

Theorem 6. *Given a constraint network and an ordering of its variables, it is strongly directionally* n*-consistent if it is dually adaptively consistent.*

Proof. We only need to prove that the network is adaptively consistent: For any variable x, its directionally relevant constraints $DR(x)$ are consistent on x. Let S be the variables involved in $DR(x)$. Consider any consistent instantiation $\bar{a}$ of $S - \{x\}$. We show that there exists $u \in D_x$ such that $(\bar{a}, u)$ satisfies constraints in $DR(x)$. Let l be the number of constraints in $DR(x)$, and let c_x be one of the tightest constraint in $DR(x)$ with tightness m_x. For any constraint $c_i \in DR(x)(i : 1..l)$, let $\bar{a}$'s support set on x under c_i be E_i. It is sufficient to show $\cap_{i \in 1..l} E_i \neq \emptyset$. We know c_x is consistent with every other m_x constraints. Hence, E_x, $\bar{a}$'s support set under c_x, intersects with every other m_x support sets of $\bar{a}$. By the set intersection lemma, $\cap_{i \in 1..l} E_i \neq \emptyset$. □

Improving Bucket Elimination on Constraint Networks For a variable x, the fact that we enforce all its directionally relevant constraints consistent on x, is described as joining (a Database operation of *natural join*) all the constraints (taken as relations) and projecting away x (and thus eliminating x) in bucket elimination. We know that both time and space complexity of the join operation is exponential to the number of constraints involved. In terms of dually adaptive consistency, if one of the constraints c_x is m-tight and m is smaller than the number of constraints of concern, we only need to join c_x and every other m constraints and then project away x. An extreme case is that if a constraint c_x is 1-tight, it is sufficient to join c_x and every other constraint of concern, and then project away x.

5 Conclusion

The theme of this paper is to study the impact of the tightness of constraints on the global consistency of a network. Specifically, the tightness of the constraints determines the level of local consistency sufficient to guarantee global consistency. Under the concept of k-consistency, to determine the local consistency ensuring global consistency, we show that it is sufficient to consider only some of the binary constraints. We also show that a weakly tight constraint network *does* need a significant number of constraints to be tight. After studying directional consistency, we propose a new type of consistency – dually adaptive consistency – which considers not only the topological structure, but also the tightness of the constraints in a network. Based on this concept, only the tightest (in a local sense) constraints or the widths of variables, depending on which are smaller, determine the local consistency ensuring global consistency.

Dually adaptive consistency immediately leads to a more efficient version of *bucket elimination* algorithm for constraint networks [1], and may be helpful where the heuristics from bucket elimination have shown some promise (e.g., [3]). Having shown that theoretically there is a close relationship between the tightness of constraints and global consistency, in the future, we will explore whether the tightness of constraints can play greater role in solving practical constraint networks.

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