Consistency and Set Intersection

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Introduction

The study of local and global consistencies is an important topic in Constraint Networks (CN). For example, there are many results which relate different levels of consistency in a CN (Freuder 1982; Van Beek and Dechter 1995; Van Beek and Dechter 1997; Jeavons, Cohen and Gyssens 1997). In this paper, we present a new framework to study consistency purely in terms of general properties of set intersection.

Given a collection of l finite sets, under what conditions is the intersection of all these l sets not empty.

The significance of such set intersection results is that they can be lifted directly to a constraint network setting to obtain consistency results. We give such a proof schema to lift these results.

An example is the well known result on set intersection – any l intervals on real numbers intersect if and only if every two of them intersect. Our work is motivated by the observation from the example that local information on the intersection of two sets implies global information on the intersection of all sets. In the study of CN, it is desirable to derive global consistency from a certain level of local consistency because we would like to have an efficient consistency method, such as that from some local consistency while attaining global consistency where possible. Along these lines, several classes of special constraints have been identified in the work of van Beek and Dechter. For example, for a CN with binary *row-convex* constraints, *path consistency* is the local consistency property sufficient to guarantee global consistency. In our framework, the above result can be directly obtained from a property of the intersection of *convex* sets.

The other contribution of this paper is to present new results on finite set intersections. Based on these results and the new framework, the consistency results in (Van Beek and Dechter 1995; Van Beek and Dechter 1997) are immediate.

Set Intersection Results

Given a collection of sets $\{E_1, E_2, \cdots, E_l\}$. Van Beek and Dechter (1995) generalized the example on real intervals given in the introduction to discrete sets.

Lemma 1 (Convex Set Intersection) (Van Beek and Dechter 1995) *Let* $D = \bigcup_{i \in 1..l} E_i$. *Assume there exists a total ordering* $\preceq$ *on* D *such that for all* i, $E_i = \{v \in D \mid \min E_i \preceq v \preceq \max E_i\}$ *(set* E_i *is* convex*).* $\bigcap_{i \in 1..l} E_i \neq \emptyset$ *if and only if for any* i *and* j, $E_i \bigcap E_j \neq \emptyset$.

Lemma 1 imposes a strong restriction on the structure of the sets such that all sets are dense under a given total ordering. Motivated by the observation in (Van Beek and Dechter 1997), we have this result on unstructured sets where the only restriction is on the cardinality of the sets.

Lemma 2 (Small Set Intersection) *Assume* E_i *is finite and* $|E_i| \leq m$ $(< l)$ *for all* i . $\bigcap_{i \in 1..l} E_i \neq \emptyset$ *if and only if every* $m + 1$ *sets intersect.*

Consider the sets: $E_1 = \{1, 3, 5\}$, $E_2 = \{5, 7\}$, $E_3 = \{3, 5, 7\}$, $E_4 = \{5, 7, 9\}$. Lemma 1 is applicable (and also lemma 2) as the sets are convex under the order $(1, 3, 5, 7, 9)$. Consider instead: $E'_1 = \{1, 9\}, E'_2 = \{3, 9\}, E'_3 = \{5, 9\}, E'_4 = \{7, 9\}$. Now the sets are not convex, lemma 1 doesn't apply, but lemma 2 is applicable.

Lemma 3 (Large Set Intersection) *Assume* E_i *is finite and* $|E_i| \geq m$ *for all* i, *and* $|\bigcup_{i \in 1..l} E_i| = d$. $\bigcap_{i \in 1..l} E_i \neq \emptyset$ *if* $l \leq \lceil d/(d - m) \rceil - 1$.

Set Intersection and Consistency

We now relate set intersection and consistency in constraint networks. A *constraint network* $\mathcal{R}$ is defined as a set of variables $\{x_1, x_2, \cdots, x_n\}$, a set of domains $\{D_1, D_2, \cdots, D_n\}$ where $\forall i,\ D_i$ is the domain of x_i, and a set of constraints $\{R_{S_1}, R_{S_2}, \cdots, R_{S_e}\}$ where $\forall i,\ S_i \subseteq \{x_1, x_2, \cdots, x_n\}$. Given a constraint R_{S_i}, a variable $x \in S_i$ and any instantiation $\bar{a}$ of $S_i - \{x\}$, the *extension set* of $\bar{a}$ to x with respect to R_{S_i} is defined as $E_{i,x}(\bar{a}) = \{b \in D_x \mid (\bar{a}, b)\ satisfies\ R_{S_i}\}$.

With the notion of extension set we have the following lemma on *k-consistency* [see (Freuder 1978) for motivations and more information].

Lemma 4 (Set Intersection and Consistency) *A constraint network* $\mathcal{R}$ *is* k*-consistent if and only if for any consistent instantiation* $\bar{a}$ *of any (k-1) distinct variables*

$Y = \{x_1, x_2, \cdots, x_{k-1}\}$, *and for any new variable* x_k, $\bigcap_{j \in 1..l} E_{i_j} \neq \emptyset$ *where* E_{i_j} *is the extension set of* $\bar{a}$ *to* x_k *with respect to* $R_{S_{i_j}}$ *where* $R_{S_{i_1}}, \ldots, R_{S_{i_l}}$ *are those constraints which involve only* x_k *and a subset of variables from* Y.

Example. We use c_{ij} to denote a constraint between variables x_i and x_j. Consider $c_{13} = \{(1,5),(1,7),(2,9)\}, c_{23} = \{(3,1),(3,5),(5,9)\}, c_{43} = \{(5,7),(5,9),(8,9)\}$. Given an instantiation $\bar{a} = (1,3,5)$ of three variables (x_1, x_2, x_4). For x_3, there are totally three constraints involving it and other instantiated variables. The extension set of $\bar{a}$ to x_3 wrt c_{13} is $\{5,7\}$ because x_1 takes value of 1 in $\bar{a}$. The other extension sets of $\bar{a}$ to x_3 are $\{1,5\}$ (from c_{23}) and $\{7,9\}$ (from c_{43}). The intersection of the three extension sets of x_3 is empty. Thus the constraint network is not 3 consistent.

The insight behind this lemma is simply a view of consistency from the perspective of set intersection. The results on set intersection, including those in section 2, can be *lifted* to give various consistency results through the following *proof schema* (thus lemma 4 can also be called the *lifting* lemma).

Proof Schema

1. (*Consistency to Set*) From a certain level of consistency *in* the constraint network, we derive intersection information on the extension sets (according to lemma 4).

2. (*Set to Set*) From the *local* intersection information of sets, information may be obtained on intersection of more sets (according to set intersection results, for example the lemmas given in section 2).

3. (*Set to Consistency*) From the new information on set intersection, higher level of consistency is obtained (again according to lemma 4).

4. (*Formulate conclusion on the consistency of the constraint network*).

Applications to Consistency

The notion of *extension set* plays the role of a bridge between restrictions to set(s) and properties of the constraint network. Set restrictions, such as *convexity and cardinality*, can be translated to properties on constraint (through the extension set), like *row-convexity, tightness* and *looseness*. See (Van Beek and Dechter 1995; Van Beek and Dechter 1997) for these properties. The proof schema can be used with lemmas 1, 2 and 3 on set intersection to obtain more direct proofs for theorems 1, 2 and 3 below.

Theorem 1 (Tightness) (Van Beek and Dechter 1997) *If a constraint network* $\mathcal{R}$ *with constraints that are m-tight and of arity at most r is strongly* $((m+1)(r-1)+1)$*-consistent, then it is globally consistent.*

Theorem 2 (Row Convex) (Van Beek and Dechter 1995) *Let* $\mathcal{R}$ *be a network of constraints whose arity at most r is strongly* $2(r-1)+1$ *consistent. If there exists an ordering of the domains* $D_1, \cdots, D_n$ *of* $\mathcal{R}$ *such that all constraints are row convex,* $\mathcal{R}$ *is globally consistent.*

Theorem 3 (Looseness) (Van Beek and Dechter 1997) *A constraint network with domains that are of size at most* d *and constraints that are* m*-loose and of arity at least* r, $r \geq 2$, *is strongly* k*-consistent, where* k *is the maximum value such that the following inequality holds,* $binomial(k-1, r-1) \leq \lceil d/(d-m) \rceil - 1$.

The above theorem statement differs slightly from the original one (Zhang and Yap (manuscript)).

We note that the consistency lemma (lemma 4) can be migrated to *relational consistency* directly from the definition of relational consistency, and the proof schema is unchanged. The set intersection results can then be directly *lifted* to consistency results for relational consistency.

Discussion and Conclusion

We have introduced a new perspective of studying consistency using the lifting lemma and properties of set intersection. The relation between set intersection and consistency can be illustrated with reference to results of a binary CN. Suppose lemma 1 applies, then all sets will intersect if we know the *local* intersection information on every *two* sets. The consistency result is that given a corresponding constraint network, local (2+1)-consistency (or path consistency) in such a restricted network implies global consistency. Lemma 2 tells us that the intersection information on *k* sets induces intersection on *all* sets, which results in an observation (theorem 1) that global consistency follows (k+1) consistency for those kinds of networks. In lemma 3, *all* large sets with cardinality ($\geq m$) simply intersect without local intersection information. Hence, a certain level of consistency depending on m is inherent in the related constraint network.

Our work suggests that more consistency results may be obtained by purely inspecting certain set intersection problems. One possible direction is to get a lower requirement on the local intersection information identified in lemma 2 by imposing some additional structure on the sets. We believe that our framework shows potential as a general technique for obtaining more results on consistency in constraint networks from a study on properties of set intersection.

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