Consistency and Set Intersection

Yuanlin Zhang and Roland H.C. Yap
National University of Singapore
3 Science Drive 2, Singapore
{zhangyl,ryap}@comp.nus.edu.sg

Abstract

We propose a new framework to study properties of consistency in a Constraint Network from the perspective of properties of set intersection. Our framework comes with a proof schema which gives a generic way of lifting a set intersection property to one on consistency. Various well known results can be derived with this framework. More importantly, we use the framework to obtain a number of new results. We identify a new class of tree convex constraints where local consistency ensures global consistency. Another result is that in a network of arbitrary constraints, local consistency implies global consistency whenever there are certain m-tight constraints. The most interesting result is that when the constraint on every pair of variables is properly m-tight in an arbitrary network, global consistency can be achieved by enforcing relational m+1-consistency. These results significantly improve our understanding of convex and tight constraints. This demonstrates that our framework is a promising and powerful tool for studying consistency.

1 Introduction

A fundamental problem in Constraint Networks is to study properties of consistency in some networks with particular constraints or structure. There have been two main approaches to this problem.

The first is to utilise topological structure of the network. For example, where the network forms a tree structure, arc consistency is sufficient to make the network globally consistent [Freuder, 1982].

Another approach is to make use of semantic properties of the constraints. For *monotone constraints*, path consistency implies global consistency [Montanari, 1974]. Dechter [1992] shows that a certain level of consistency in a network whose domains are of limited size ensures global consistency. Van Beek and Dechter [1995] generalize monotone constraints to a larger class of *row convex constraints*. Later, they [1997] study the consistency inside a network with tight and loose constraints.

In this paper, we present a new framework[1] which unifies well known results mainly along the second approach (including [van Beek and Dechter, 1995; 1997]). The power of this framework is that it allows the study of the relationship between local and global consistency from the perspective of *set intersection*.

For example, one property of set intersection is that if every pair (*2*) of tree convex sets intersect, the whole collection of these sets also intersect. The main point is that local information on intersection of every pair of sets gives global information on intersection of all sets. Intuitively, this can be related to getting global consistency from local consistency. Our framework enables us to lift the result on tree convex sets to the following result on consistency. For a binary network of tree convex constraints, (*2*+1)-consistency (path consistency) implies global consistency.

Properties of the intersection of *tree convex* sets, *small* sets, *large* sets are presented in section 3. Section 4 develops the framework which consists of a lifting lemma together with a proof schema which gives a generic way of using the lifting lemma to get consistency results from properties of set intersection. One benefit of the framework lies in that it provides a modular way to greatly simplify the understanding and proof of results on consistency. This benefit is considerable as often the proof of many existing results is complex and “hard-wired”.

We demonstrate the power of the framework by showing three new consistency results as well as a number of well known results. The first is a generalization of global consistency of row convex constraints to a network of tree convex constraints which is presented in section 5. The second is a new result on global consistency on *weakly tight* networks in section 6. These networks only require certain constraints to be m-tight rather than all constraints as shown in [van Beek and Dechter, 1997]. The final result is presented in section 8. This is most interesting because we can make certain networks globally consistent by enforcing relational consistency, but this may not be achievable in previous work (for example [van Beek and Dechter, 1997]).

[1] Our work can be related to the first approach in those cases where the network topology leads to some set intersection property.

2 Preliminaries

A *constraint network* $\mathcal{R}$ is defined as a set of variables $N = \{x_1, x_2, \cdots, x_n\}$; a set of finite domains $D = \{D_1, D_2, \cdots, D_n\}$ where domain D_i, for all $i \in 1..n$, is a set of values that variable x_i can take; and a set of constraints $C = \{c_{S_1}, c_{S_2}, \cdots, c_{S_e}\}$ where S_i, for all $i \in 1..n$, is a subset of $\{x_1, x_2, \cdots, x_n\}$ and each constraint c_{S_i} is a relation defined on domains of all variables in S_i. The *arity* of constraint c_{S_i} is the number of variables in S_i.

An instantiation of variables $Y = \{x_1, \cdots, x_j\}$ is denoted by $\bar{a} = (a_1, \cdots, a_j)$ where $a_i \in D_i$ for $i \in 1..j$. An *extension* of $\bar{a}$ to a variable $x (\notin Y)$ is denoted by $(\bar{a}, u)$ where $u \in D_x$. An instantiation of a set of variables Y is *consistent* if it satisfies all constraints in $\mathcal{R}$ which don't involve any variable outside Y.

A constraint network $\mathcal{R}$ is *k-consistent* if and only if for any consistent instantiation $\bar{a}$ of any distinct $k-1$ variables, and for any new variable x, there exists $u \in D_x$ such that $(\bar{a}, u)$ is a consistent instantiation of the k variables. $\mathcal{R}$ is *strongly* k-consistent if and only if it is j-consistent for all $j \leq k$. A strongly n-consistent network is called *globally consistent*.

3 Properties on Set Intersection

The set intersection property which we are concerned with is:

> Given a collection of l finite sets, under what conditions is the intersection of all l sets not empty?

We use $\mathcal{S}$ to denote the collection of l sets: $\{E_1, E_2, \cdots, E_l\}$, and U the union of all the sets in $\mathcal{S}$, that is $U = \bigcup_{i \in 1..l} E_i$.

This property may not make sense for collections of arbitrary sets. Here, we study sets with two restrictions.

3.1 Sets with Convexity Restrictions

Definition 1 *Given a set U and a tree $\mathcal{T}$ with vertices U. A set $A \subseteq U$ is* tree convex *iff there exists a subtree, with only vertices A, of $\mathcal{T}$. The sets in $\mathcal{S}$ are* tree convex *if there is a tree on U under which every set in $\mathcal{S}$ is tree convex.*

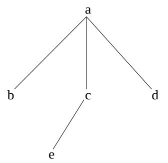

Figure 1: A tree with nodes a, b, c, d, e

Example. Consider a set $U = \{a, b, c, d, e\}$ and a tree in Fig 1. The subset $\{a, b, c, d\}$ is tree convex. So is the set $\{b, a, c, e\}$ since the elements in the set consists of a subtree. However, $\{b, c, e\}$ is not tree convex for they does not form a subtree of the given tree.

Consider $\mathcal{X} = \{\{1, 9\}, \{3, 9\}, \{5, 9\}\}$. A tree can be constructed on $\{1, 3, 5, 9\}$ with 9 being the root and 1, 3, 5 being its children. Each set in $\mathcal{X}$ covers the nodes of exactly one branch of the tree. Hence the sets in $\mathcal{X}$ are tree convex. □

Tree convex sets have the following intersection property.

Lemma 1 (Tree convex Sets Intersection) *Assume the sets in $\mathcal{S}$ are tree convex. $\bigcap_{i \in 1..l} E_i \neq \emptyset$ iff for every $E_i, E_j \in \mathcal{S}$, $E_i \bigcap E_j \neq \emptyset$.*

Proof. Let $\mathcal{T}$ be a tree such that there exists a subtree T_i for each $E_i \in \mathcal{S}$. We can regard $\mathcal{T}$ as a rooted tree and thus every T_i ($i \in 1..l$) can be regarded as a rooted tree whose root is exactly the node nearest to the root of $\mathcal{T}$. Let r_i denote the root of T_i for $i \in 1..l$.

To prove $\bigcap_{i \in 1..l} E_i \neq \emptyset$, we want to show the intersection of the trees $\{T_i \mid i \in 1..l\}$ is not empty. The following proposition on subtrees, whose proof is omitted, is necessary in our main proof.

> **Proposition 1** *Let the root of $\mathcal{T}$ be r. Given two subtrees T_1 and T_2 of $\mathcal{T}$ with roots r_1 and r_2 respectively. Let r_1 be not closer to r than r_2, and T the intersection of T_1 and T_2. r_1 is the root of T if T is not empty.*

Let $T = \bigcap_{i \in 1..l} T_i$. We are ready now to prove our main result $T \neq \emptyset$. We select a tree $T_{\max}$ from $T_1, T_2, \cdots, T_l$ such that its root $r_{\max}$ is the farthest away from r of $\mathcal{T}$ among the roots of the concerned trees. In terms of proposition 1, that $T_{\max}$ intersect with every other trees implies that $r_{\max}$ is a node of every T_i ($i \in 1..l$). Therefore, $r_{\max} \in T$. □

Tree convexity on a collection of sets imposes a strong restriction on the relationship among these sets in the sense that each set forms part of a tree. If the tree is a chain, it can be thought of as a total order on the elements of all the sets. This leads to the notion of convex sets upon which the well-known row-convex constraints are defined.

Definition 2 *Given a set U and a total ordering "$\preceq$" on it, a set $A \subset U$ is* convex *if the elements in it are consecutive under the ordering, that is*

$$A = \{v \in U \mid \min A \preceq v \preceq \max A\}.$$

Given $\mathcal{S}$ and U, the sets in $\mathcal{S}$ are convex *if there is a total ordering on U such that every set in $\mathcal{S}$ is convex under the ordering.*

Example. The set of real numbers between 1 and 2 is convex under the usual ordering of numbers. $\{1, 9\}$ and $\{3, 9\}$ are convex with a total ordering $1 \preceq 9 \preceq 3$. However, $\{1, 9\}, \{3, 9\}$ and $\{5, 9\}$ are not convex under any total ordering on $\{1, 3, 5, 9\}$. □

The following result on convex sets, a set-based version of van Beek and Dechter's [1995, lemma 3.1], is immediate.

Corollary 1 (Convex Sets Intersection) *Assume the sets in $\mathcal{S}$ are convex under a total ordering on U. $\bigcap_{i \in 1..l} E_i \neq \emptyset$ iff for any $E_i, E_j \in \mathcal{S}$, $E_i \bigcap E_j \neq \emptyset$.*

3.2 Sets with Cardinality Restrictions

The following results are on arbitrary sets where the only restriction is on the size of sets.

Lemma 2 (Small Set Intersection) *Given a collection of sets $\mathcal{S}$. Assume there is a set $E \in \mathcal{S}$ such that $|E| \leq m$.*

$$\bigcap_{i \in 1..l} E_i \neq \emptyset$$

iff the intersection of any $m+1$ sets from $\mathcal{S}$ is not empty.

Proof. The necessary condition is immediate.

The sufficient condition is proven by induction on l. It is obviously true when $l \le m+1$. Assuming that $k\ (> m)$ sets intersect, we show that any $k+1$ sets intersect. Without loss of generality, the subscripts of the $k+1$ sets are numbered from 1 to $k+1$ and let $|E_1| \le m$. Let A_i be the intersection of all the $k+1$ sets except E_i:

$$A_i = E_1 \cap \cdots \cap E_{i-1} \cap E_{i+1} \cap \cdots \cap E_{k+1},\ for\ 1 < i \le k+1.$$

If $A_i \cap A_j \ne \emptyset$ for some $i, j \in 2..k+1, i \ne j$,

$$\bigcap_{i \in 1..k+1} E_i = A_i \cap A_j \ne \emptyset.$$

Assume the contrary that $A_i \cap A_j = \emptyset$ for all distinct i and j. According to the construction of A_i's,

$$E_1 \supseteq \bigcup_{i \in 2..k+1} A_i,$$

and $|A_i| \ge 1$ by the induction assumption. Henceforth,

$$|E_1| \ge \sum_{i \in 2..k+1} |A_i| \ge k > m$$

which contradicts $|E_1| \le m$. □

We have an immediate corollary of the lemma when all sets of concern have at most m elements.

Corollary 2 (Small Sets Intersection) *For any $E_i \in \mathcal{S}$, assume E_i is finite and $|E_i| \le m\ (< l)$. $\bigcap_{i \in 1..l} E_i \ne \emptyset$ iff the intersection of any $m+1$ sets from $\mathcal{S}$ is not empty.*

Another special case is that some set has only *one* element.

Corollary 3 (Singleton Set Intersection) *Given a collection of sets $\mathcal{S}$. Assume there is a set $E \in \mathcal{S}$ such that $|E| = 1$. $\bigcap_{i \in 1..l} E_i \ne \emptyset$ iff all sets mutually intersect.*

Motivated by [van Beek and Dechter, 1997, lemma 4.1 in page 561], we consider the intersection of sets, each of which has at least some number of elements.

Lemma 3 (Large Sets Intersection) *For all $E_i \in \mathcal{S}$, assume E_i is finite and $|E_i| \ge m$. Let $|\bigcup_{i \in 1..l} E_i| = d$. if $l \le \lceil d/(d-m) \rceil - 1$, then $\bigcap_{i \in 1..l} E_i \ne \emptyset$.*

Proof. Let $U = \bigcup_{i \in 1..l} E_i$, and $A_i = U - E_i$ for all $i \le l$. It is immediate that

$$\bigcup_{i \in 1..l} A_i \subseteq U.$$

We know

$$|\bigcup_{i \in 1..l} A_i| \le \sum_{i \in 1..l} |A_i|.$$

For $|A_i| \le d - m$, we have

$$\sum_{i \in 1..l} |A_i| \le \sum_{i \in 1..l} (d-m) = l(d-m) < d.$$

Hence, $\bigcup_{i \in 1..l} A_i$ is a proper subset of U. There exists $x \in U$ such that $x \notin A_i$ for all $i \le l$, which implies that $x \in E_i$ for all $i \le l$. □

4 Set Intersection and Consistency

In this section we relate set intersection with consistency in constraint networks.

Underlying the concept of consistency is whether an instantiation of some variables can be extended to a new variable such that all relevant constraints to the new variable are satisfied. A *relevant* constraint to a variable x_i is a constraint where only x_i is uninstantiated and the others are instantiated. Each relevant constraint allows a set (possibly empty) of values for the new variable. This set is called *extension set* below. The satisfiability of all relevant constraints depends on whether the intersection of their extension sets is non-empty (see lemma 4).

Definition 3 *Given a constraint c_{S_i}, a variable $x \in S_i$ and any instantiation $\bar{a}$ of $S_i - \{x\}$, the* extension set *of $\bar{a}$ to x with respect to c_{S_i} is defined as*

$$E_{i,x}(\bar{a}) = \{b \in D_x \mid (\bar{a}, b)\ satisfies\ c_{S_i}\}.$$

An extension set is trivial *if it is empty; otherwise it is* non-trivial.

Throughout this paper, it is often the case that an instantiation $\bar{a}$ of $S - \{x\}$ is already given where $S - \{x\}$ is a superset of $S_i - \{x\}$. Let $\bar{b}$ be the instantiation obtained by restricting $\bar{a}$ to the variables only in $S_i - \{x\}$. For ease of presentation, we continue to use $E_{i,x}(\bar{a})$, rather than $E_{i,x}(\bar{b})$, to denote the extension of $\bar{b}$ to x under constraint c_{S_i}. Where the context is clear, we may omit some of the three parameters i, $\bar{a}$ and x.

Example. Consider the network with variables $\{x, x_1, x_2, x_4, x_5\}$:

$$\begin{array}{lcll} c_{S_1} & = & \{(a,b,d),(a,b,a)\}, & S_1 = \{x_1, x_2, x\}; \\ c_{S_2} & = & \{(b,a,d),(b,a,b)\}, & S_2 = \{x_2, x_4, x\}; \\ c_{S_3} & = & \{(b,d),(b,c)\}, & S_3 = \{x_2, x\}; \\ c_{S_4} & = & \{(b,a,d),(b,a,a)\}, & S_4 = \{x_2, x_5, x\}; \end{array}$$
$$D_1 = D_4 = D_5 = \{a\}, D_2 = \{b\}, D_x = \{a, b, c, d\}.$$

Let $\bar{a} = (a, b, a)$ be an instantiation of variables $Y = \{x_1, x_2, x_4\}$. The relevant constraints to x are c_{S_1}, c_{S_2}, and c_{S_3}. c_{S_4} is not relevant since it has two uninstantiated variables. The extension sets of $\bar{a}$ to x with respect to the relevant constraints are:

$$E_1(\bar{a}) = \{d, a\}, E_2(\bar{a}) = \{d, b\}, E_3(\bar{a}) = \{d, c\}.$$

The intersection of the above extension sets is not empty, implying that $\bar{a}$ can be extended to satisfy all relevant constraints c_{S_1}, c_{S_2} and c_{S_3}.

Let $\bar{a} = (b, b)$ be an instantiation of $\{x_2, x\}$. $E_{1,x_1}(\bar{a}) = \emptyset$ and it is trivial. In other words, when an instantiation has a trivial extension set, it can not be extended to satisfy the constraint of concern. □

The relationship between *k-consistency* and set intersection is characterized by the following lemma which is a direct consequence of the definition of *k-consistency*.

Lemma 4 (Set Intersection and Consistency; lifting) *A constraint network $\mathcal{R}$ is k-consistent if and only if for any consistent instantiation $\bar{a}$ of any $(k-1)$ distinct variables $Y = \{x_1, x_2, \cdots, x_{k-1}\}$, and any new variable x_k,*

$$\bigcap_{j \in 1..l} E_{i_j} \ne \emptyset$$

where E_{i_j} is the extension set of $\bar{a}$ to x_k with respect to $c_{S_{i_j}}$, and $c_{S_{i_1}}, \cdots, c_{S_{i_l}}$ are all relevant constraints.

The insight behind this lemma is to see consistency from the perspective of set intersection.

Example. Consider again the previous example. We would like to check whether the network is 4-consistent. Consider the instantiation $\bar{a}$ of Y again. This is a trivial consistent instantiation since the network doesn't have a constraint among the variables in Y. To extend it to x, we need to check the first three constraints. The extension is feasible because the intersection of E_1, E_2, and E_3 is not empty. We show the network is 4-consistent, by exhausting all consistent instantiations of any three variables. Conversely, if we know the network is 4-consistent, we can immediately say that the intersection of the three extension sets of $\bar{a}$ to x is not empty. □

The usefulness of this lemma is that it allows consistency information to be obtained from the intersection of extension sets, and vice versa. Using this view of consistency as set intersection, some results on set intersection properties, including all those in section 3, can be *lifted* to get various consistency results for a constraint network through the following *proof schema*.

Proof Schema

1. (*Consistency to Set*) From a certain level of consistency *in* the constraint network, we derive information on the intersection of the extension sets according to lemma 4.
2. (*Set to Set*) From the *local* intersection information of sets, information may be obtained on intersection of more sets.
3. (*Set to Consistency*) From the new information on intersection of extension sets, higher level of consistency is obtained according to lemma 4.
4. (*Formulate conclusion on the consistency of the constraint network*). □

Given the proof schema, lemma 4 is also called the *lifting* lemma.

In the following sections, we demonstrate how the set intersection properties and the proof schema are used to obtain new and also well known results on consistency of a network.

5 Application I: Global Consistency of Tree Convex Constraints

The notion of *extension set* plays the role of a bridge between the restrictions to set(s) and properties of special constraints. The sets in lemma 1 are restricted to be tree convex. If all extension sets of a constraint is tree convex, we call this constraint *tree convex*.

Definition 4 *A constraint c_S is* tree convex *with respect to x if and only if the sets in*

$$A = \{E_{S,x} \mid E_{S,x} \textit{ is a non-trivial extension of some instantiation of } S - \{x\}\}$$

are tree convex. c_S is tree convex *if it is tree convex wrt every $x \in S$ under a common tree on the union of domains involved.*

$\{(1,1),(1,9),(3,3),(3,9),(5,5),(5,9),(9,1),(9,9)\}$ is an example of a tree convex constraint but it is not row convex.

Definition 5 *A constraint network is* tree convex *if and only if all constraints are tree convex under a common tree on the union of all domains.*

Convex sets naturally give rise to convex constraints which is a special case of tree convex constraints.

Definition 6 *A constraint c_S is* row convex *with respect to x if and only if the sets in*

$$A = \{E_{S,x} \mid E_{S,x} \textit{ is a non-trivial extension of some instantiation of } S - \{x\}\}$$

are convex. It is row convex if under a total ordering on the union of involved domains, it is row convex wrt every $x \in S$. A constraint network is row convex *iff all constraints are row convex under a common total ordering on the union of all domains.*

The consistency results on these networks can be derived from the property of set intersection using the proof schema. We obtain the main result of this section.

Theorem 1 (tree convexity) *Let $\mathcal{R}$ be a network of constraints with arity at most r and $\mathcal{R}$ be strongly $2(r-1)+1$ consistent. If $\mathcal{R}$ is tree convex then it is globally consistent.*

Proof. The network is strongly $2(r-1)+1$ consistent by assumption. We prove by induction that the network is k consistent for any $k \in \{2r, \cdots, n\}$.

Consider any instantiation $\bar{a}$ of any $k-1$ variables and any new variable x. Let the number of relevant constraints be l. For each relevant constraint there is one extension set of $\bar{a}$ to x. So we have l extension sets. If the intersection of all l sets is not empty, we have a value for x such that the extended instantiation satisfies all relevant constraints.

(*Consistency to Set*) Consider any two of the l extension sets: E_1 and E_2. The two corresponding constraints involve at most $2(r-1)+1$ variables since the arity of a constraint is at most r and each of the two constraints has x as a variable. According to the consistency lemma, that $\mathcal{R}$ is $2(r-1)+1$-consistent implies that the intersection of E_1 and E_2 is not empty.

(*Set to Set*) Since all relevant constraints are tree convex under the given tree, the extension sets of $\bar{a}$ to x are tree convex. Henceforth, the fact that every two of the extension sets intersect shows that the intersection of all l extension sets is not empty, in terms of the *tree convex sets intersection* lemma.

(*Set to Consistency*) From the consistency lemma, we have that $\mathcal{R}$ is k-consistent. □

Since a row convex constraint is tree convex, we have the following result.

Corollary 4 (row convexity) [van Beek and Dechter, 1995] *Let $\mathcal{R}$ be a network of constraints with arity at most r and $\mathcal{R}$ be strongly $2(r-1)+1$ consistent. If there exists an ordering of the domains $D_1, \cdots, D_n$ of $\mathcal{R}$ such that $\mathcal{R}$ is row convex, $\mathcal{R}$ is globally consistent.*

This can also be proved directly by lifting corollary 1.

6 Application II: Global Consistency on Weakly Tight Networks

In this section, we study networks with some tight constraints. The *m-tight* property of a constraint is related to the cardinality of the extension set in the following way.

Definition 7 *A constraint c_{S_i} is* m-tight *with respect to $x \in S_i$ iff for any instantiation $\bar{a}$ of $S_i - \{x\}$,*

$$|E_{i,x}| \leq m \textit{ or } |E_{i,x}| = |D_x|.$$

A constraint c_{S_i} is m-tight *iff it is m-tight with respect to every $x \in S_i$.*

Definition 8 *A constraint network is* weakly *m-tight at level k iff for every set of variables $\{x_1, \cdots, x_l\} (k \leq l \leq n)$ and a new variable, there exists an m-tight constraint in the relevant constraints after the instantiation of the l variables.*

The small set intersection lemma gives the following theorem.

Theorem 2 (Weak Tightness) *If a constraint network $\mathcal{R}$ with constraints of arity at most r is strongly $((m+1)(r-1)+1)$-consistent and weakly m-tight at level $((m+1)(r-1)+1)$, it is globally consistent.*

The proof is similar to the theorem on tree convexity.

Immediately we have the following result which is a main result in [van Beek and Dechter, 1997].

Corollary 5 (Tightness) [van Beek and Dechter, 1997] *If a constraint network $\mathcal{R}$ with constraints that are m-tight and of arity at most r is strongly $((m+1)(r-1)+1)$-consistent, then it is globally consistent.*

This result can of course be directly lifted from corollary 2. Corollary 5 requires every constraint to be m-tight. The weak tightness theorem on the other hand does not require all constraints to be m-tight. The following example illustrates this difference.

Example. For a weakly m-tight network, we are interested in its topological structure. Thus we have omitted the domains of variables here. Consider a network with five variables labelled $\{1, 2, 3, 4, 5\}$. In this network, for any pair of variables and for any three variables, there is a constraint. Assume the network is already strongly 4-consistent.

Since the network is already strongly 4-consistent, we can simply ignore the instantiations with less than 4 variables. This is why we introduce the level at which the network is weakly m-tight. The interesting level here is 4. Table 1 shows the relevant constraints for each possible extension of four instantiated variables to the other one. In the first row, $1234 \rightarrow 5$ stands for extending the instantiation of variables $\{1, 2, 3, 4\}$ to variable 5. Entries in its second column denote a constraint. For example, 125 denotes c_{125}. If the constraints on $\{1, 2, 5\}$ and $\{1, 3, 4\}$ (suffixed by * in the table) are m-tight, the network is weakly m-tight at level 4. Alternatively, if the constraints $\{1, 5\}$, $\{2, 3\}$ and $\{3, 4\}$ (suffixed by +) are m-tight, the network will also be weakly m-tight. However, the tightness corollary requires all binary and ternary constraints to be m-tight. The weak m-tightness theorem needs significantly less constraints to be m-tight. Further results are in section 8. □

extension	relevant constraints				
$1234 \rightarrow 5$	125*,	135 ,	145 ,	235,	245,
	345,	15+,	25 ,	35 ,	45
$2345 \rightarrow 1$	231 ,	241 ,	251*,	341,	351,
	451,	21 ,	31 ,	41 ,	51+
$3451 \rightarrow 2$	132 ,	142 ,	152*,	342,	352,
	452,	12 ,	32+,	42 ,	52
$4512 \rightarrow 3$	123 ,	143*,	153 ,	243,	253,
	453,	13 ,	23+,	43 ,	53
$5123 \rightarrow 4$	124 ,	134*,	154 ,	234,	254,
	354,	14 ,	24 ,	34+,	54

Table 1: Relevant constraints in extending the instantiation of four variables to the other one

7 Application III: Constraint Looseness

To make the presentation complete, we list the result on looseness of constraints by van Beek and Dechter [1997] (see also [Zhang and Yap, 2003]) as an instance of our framework.

The *m-loose* property of a constraint is related to the cardinality of extension set in the following way.

Definition 9 *A constraint c_{S_i} is* m-loose *with respect to $x \in S_i$ iff for any instantiation $\bar{a}$ of $S_i - \{x\}$, $|E_{i,x}| \geq m$. A constraint c_{S_i} is* m-loose *iff it is m-loose with respect to every $x \in S_i$.*

The *large set intersection* lemma is lifted to the following result on *constraint looseness*.

Theorem 3 (Looseness) *Given a constraint network with domains that are of size at most d and constraints that are m-loose and of arity at least r, $r \geq 2$. It is strongly k-consistent, where k is the maximum value such that $binomial(k-1, r-1) \leq \lceil d/(d-m) \rceil - 1$.*

8 Application IV: Making Weakly Tight Networks Globally Consistent

In this section, relational consistency will be used to make a constraint network globally consistent.

Definition 10 [van Beek and Dechter, 1997] *A constraint network is* relationally m-consistent *iff given (1) any m distinct constraints $c_{S_1}, \cdots, c_{S_m}$, and (2) any $x \in \cap_{i=1}^{m} S_i$, and (3) any consistent instantiation $\bar{a}$ of the variables in $(\cup_{i=1}^{m} S_i - \{x\})$, there exists an extension of $\bar{a}$ to x such that the extension is consistent with the m relations. A network is* strongly *relationally m-consistent if it is relationally j-consistent for every $j \leq m$.*

All the results on *small set intersection* and *tree convex set intersection* in section 3 can be rephrased by using relational consistency. For example, a new version of weak tightness is shown below. Its proof can be easily obtained following the proof schema.

Theorem 4 (Weak Tightness) *If a constraint network $\mathcal{R}$ of constraints with arity of at most r is strongly relationally $(m+1)$-consistent and weakly m-tight at level of $(m+1)(r-1)+1$, it is globally consistent.*

Consider the weak m-tightness theorem 2 in section 6 based on local k-consistency. A weakly m-tight network in general may not have the level of consistency required by the theorem. To obtain global consistency, one may try to enforce this level of consistency. However, this may result in new constraints with higher arity in the network. The new constraints may in turn require a higher level of consistency to ensure global consistency. Therefore it is difficult to predict an exact level of consistency to enforce on the network to make it globally consistent.

Using relational consistency, it is possible to obtain global consistency by enforcing local consistency on the network. In order to achieve our main result we need a stronger version of m-tightness — *proper* m-tightness.

Definition 11 *A constraint c_{S_i} is* properly m-tight *with respect to $x \in S_i$ iff for any instantiation $\bar{a}$ of $S_i - \{x\}$,*

$$|E_{i,x}| \leq m.$$

A constraint c_{S_i} is properly m-tight *iff it is properly m-tight with respect to every $x \in S_i$.*

A constraint is m-tight if it is properly m-tight. The converse may not be true.

A *weakly properly m-tight network* is defined by replacing "m-tight" with "properly m-tight" in definition 8 (section 6).

We have the following observation on the weak m-tightness and weak proper m-tightness of a network.

Proposition 2 *A constraint network is weakly m-tight and weakly properly m-tight respectively if the constraint between every* two *variables in the network is m-tight and properly m-tight respectively.*

It is easy to verify that the *proper m-tightness* of the binary constraints is preserved during the procedure to enforce certain level of consistency in the network. So we have the main result of this section.

Theorem 5 (Weak Proper-Tightness) *Given a constraint network whose constraint on every two variables is properly m-tight. It is globally consistent after it is made relationally $m + 1$-consistent.*

This theorem follows immediately from the discussion above and theorem 4. The implication of this theorem is that as long as we have certain properly m-tight constraints on certain combinations of variables, the network can be made globally consistent by enforcing relational $m+1$-consistency.

9 Summary

We demonstrate how to infer properties of consistency on a network purely by making use of set intersection properties. In addition to the results shown here, some other results can also be obtained by the lifting lemma. For example, the work of David [1993] can be obtained by lifting the singleton set corollary 3. The work of Faltings and Sam-Haroud [1996] is on convex constraint networks in continuous domains and the idea there is to lift Helly's theorem on intersection of convex sets in Euclidean spaces.

We show a number of new consistency results which we believe are significant progress to convexity and tightness of constraints since van Beek and Dechter's work [1995; 1997]. We identify a class of tree convex constraints which is a superset of row convex constraints [van Beek and Dechter, 1995]. In a network of tree convex constraints, global consistency is ensured by a certain level of local consistency.

We show that *in a network of arbitrary constraints, local consistency implies global consistency whenever there are m-tight constraints on certain variables* (e.g. theorem 2). However, when the network does not have the required local consistency, global consistency may not be simply obtained by enforcing such level of local consistency. A surprising result is that as long as the constraint between every pair of variables is *properly* m-tight in an arbitrary network, global consistency can be achieved by enforcing a certain level of *relational* consistency (theorem 5). In previous work (e.g. [van Beek and Dechter, 1997]), all constraints are required to be m-tight which may be violated by newly introduced constraints in the process of enforcing the intended relational consistency.

A promising line of work is to find more properties under which a network is weakly properly m-tight. Another obvious direction is to find other classes of sets with intersection properties which will likely give useful consistency results.

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