Erratum: P. van Beek and R. Dechter's Theorem on Constraint Looseness and Local Consistency

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In order to prune the search space of a constraint network [Mackworth 1977], local (certain level of) consistency, for example arc consistency or path consistency, is enforced on the constraint network of the problem. Van Beek and Dechter observe in their theorem 4.2 [van Beek and Dechter 1997, page 562] that some inherent level of consistency may be readily present for some special constraint networks. Lower level of consistency enforcing does not produce any pruning of the search space of those constraint networks at all, and thus a higher level of consistency than the inherent one may be necessary to prune the search space. We notice that theorem 4.2 may overestimate the level of inherent consistency. Here is a counterexample. The same notations as those in [van Beek and Dechter 1997] are employed in the following presentation.

Example 1. Let $D = \{a_1, a_2, a_3\}$. Define a constraint network $\mathcal{R}$ with variables $\{x_1, x_2, x_3, x\}$, domains being D, and constraints

$$\begin{aligned} \{\ \ & R_{S_1} = D \otimes D \otimes D - \{(a, a, a)\}, \\ & R_{S_2} = D \otimes D \otimes D - \{(a, a, b)\}, \\ & R_{S_3} = D \otimes D \otimes D - \{(a, a, c)\} \quad \} \end{aligned}$$

where $S_1 = \{x_1, x_2, x\}$, $S_2 = \{x_2, x_3, x\}$ and $S_3 = \{x_1, x_3, x\}$.

It is easy to verify that in $\mathcal{R}$, every constraint is 2-loose and the arity of constraints is $r = 3$.

According to theorem 4.2, $\mathcal{R}$ is strongly 4-consistent because the minimum k to satisfy

$$binomial(k-1, 2) \geq \lceil d/(d-m) \rceil - 1 = \lceil 3/(3-2) \rceil - 1 = 2$$

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is 4. Consider the consistent instantiation (a, a, a) of variables $\{x_1, x_2, x_3\}$. Its extensions to x with respect to R_{S_1}, R_{S_2}, and R_{S_3} are $\{b, c\}$, $\{a, c\}$, and $\{a, b\}$ respectively. Their intersection is empty, indicating that (a, a, a) is not extensible to x. Hence, $\mathcal{R}$ is not strongly 4-consistent. □

In the proof of theorem 4.2, the minimum c is found such that

$$binomial(c, r-1) \geq l.$$

The problem there is that this c may not make $binomial(c, r-1)$ greater than $\lceil d/(d-m) \rceil - 1$ although $l \leq \lceil d/(d-m) \rceil - 1$.

A lower bound of the inherent level of consistency is given by the following revised version of theorem 4.2.

THEOREM 1. *A constraint network with domains that are of size at most d and relations that are m-loose and of arity at least r, $r \geq 2$, is strongly k-consistent, where k is the maximum value such that the following inequality holds,*

$$binomial(k-1, r-1) \leq \lceil d/(d-m) \rceil - 1.$$

PROOF. Let $X = \{x_1, x_2, \cdots, x_{K-1}\}$ be a set of any $K-1$ variables where $K \leq k$, $\bar{a}$ an instantiation of the variables in X that is consistent with the constraint network, and x_K be an additional variable. Let l be the number of constraints involving x_K and only a subset of variables of X. It can be shown that

$$l \leq binomial(k-1, r-1) \leq \lceil d/(d-m) \rceil - 1.$$

So, according to lemma 4.1 in [van Beek and Dechter 1997], there exists an extension of $\bar{a}$ to x_K that is consistent to the constraint network. □

Consider the network $\mathcal{R}$ in example 1. Theorem 1 implies that $\mathcal{R}$ is strongly 3-consistent since the maximum k to satisfy

$$binomial(k-1, 2) \leq \lceil d/(d-m) \rceil - 1 = 2$$

is 3. It is not difficult to verify that $\mathcal{R}$ is strongly 3-consistent.

A tighter lower bound of the inherent level of consistency may be obtained by introducing the concept of extension degree [Zhang 2002].

Definition. Given a constraint network $\mathcal{R}$ and a set of variables $Y \subseteq N$ where N denotes the set of variables in $\mathcal{R}$. The *involvement degree* of a variable $x \in (N - Y)$ with respect to Y is the number of constraints which involves only x and some variables in Y. The *extension degree* of Y is the maximum involvement degree of all variables in $N - Y$. The *extension degree* of a positive number $k < n$ is the maximum extension degree of all subsets (of N) with k variables.

THEOREM 2. *A constraint network $\mathcal{R}$ with domains that are of size at most d and constraints that are m-loose, is strongly k-consistent, where k is the least number the extension degree of which is greater than $\lceil d/(d-m) \rceil - 1$, or $k = n$ if the extension degree of any number from 1 to $n-1$ is less than or equal to $\lceil d/(d-m) \rceil - 1$.*

It is a direct consequence of lemma 4.1.

Example 2. To see the difference between theorem 1 and theorem 2, we construct a new network $\mathcal{R}'$ from $\mathcal{R}$ in example 1 by removing constraint R_{S_3}.

Consider a set of variables $Y = \{x_1, x_2, x_3\}$. The involvement degree of x with respect to Y in $\mathcal{R}'$ is two. The extension degree of Y is two. The extension degree of 3 is also two since the extension degree of any subset of $\{x_1, x_2, x_3, x\}$ with 3 variables is at most two. The extension degrees of 1 and 2 are zero and one respectively. According to theorem 2, $\mathcal{R}'$ is strongly 4-consistent because all extension degrees are not greater than $\lceil d/(d-m) \rceil - 1 (= 2)$. However, it is only strongly 3-consistent according to theorem 1.

It can be verified that $\mathcal{R}'$ is strongly 4-consistent. □

Remark. Not affected by the bug in theorem 4.2, the corollary 4.3 on binary constraint network [van Beek and Dechter 1997, page 562] of theorem 4.2 is still correct.

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